ASYNCHRONOUS PROBABILISTIC COUPLINGS

in Higher-Order Separation Logic

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One-time pad

We consider a real and an ideal implementation of the OTP encryption

$$\mathsf{real} \triangleq \lambda(m \colon \mathsf{bool}).\,\mathsf{let}\; k = \mathsf{flip}\;\; \mathsf{in}\; (k \;\mathsf{xor}\; m)$$

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This is captured by contextual equivalence

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Multiple examples of this in Iris, e.g. ReLoC

Clutch in a nutshell

In this work we develop Clutch[1], which consists of

- A probabilistic operational semantics for sequential probabilistic languages
- A unary coupling-based WP to prove relations between probabilistic programs
- A ReLoC-style logical relation to prove contextual refinement of probabilistic programs
- A ghost resource to reason about samples that happen in the future

[1] https://github.com/logsem/clutch

Structure of Clutch

ReLoC
ReLoC logical refinement
ReLoC type interpretation
HeapLang WP rules
HeapLang
Iris WP
Iris base language
Iris base logic

Clutch
ReLoC + Clutch logical refinement
ReLoC + Clutch type interpretation
Iris + Clutch WP rules
$\mathbf{F}^{\text{rand}}_{\mu,\text{ref}}$
Clutch WP
Clutch base language
Iris base logic

A proof in Clutch

$$\dfrac{}{\vDash \lambda m.\mathsf{let}\ k = \mathsf{flip}\ \mathsf{in}\ (k\ \mathsf{xor}\ m) \precsim \lambda m.\ \mathsf{flip} : \mathsf{bool} \to \mathsf{bool}}$$

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$$\frac{f : \mathbb{B} \to \mathbb{B} \text{ bijection} \qquad \forall b\colon \mathbb{B}.\ \Delta \vDash_{\mathcal{E}} K[b] \precsim K'[f(b)] : \tau}{\Delta \vDash_{\mathcal{E}} K[\,\mathsf{flip}\,] \precsim K'[\,\mathsf{flip}\,] : \tau}\ \text{COUPL}$$

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```
Lemma real_ideal_rel :
  ⊢ REL real << ideal : lrel_bool → lrel_bool.
Proof.
  rel_arrow_val.
  iIntros (msg1 msg2) "Hmsg".
  rel_pures_l. rel_pures_r.
  foldxor.
  iDestruct "Hmsg" as "[%b [-> ->]]".
  rel_apply (refines_couple_flip_flip (xor_sem b)).
  iIntros (k).
  rel_pures_l.
  foldxor.
  iApply xor_xor_sem.
Qed.
```

Operational semantics of probabilistic languages

A probabilistic sequential language

We introduce $\mathbf{F}^{\text{rand}}_{\mu,\text{ref}}$: sequential fragment of HeapLang plus sampling

$$\begin{aligned}\tau \in \text{Type} ::= {}& \alpha \mid \text{unit} \mid \text{bool} \mid \text{nat} \mid \text{int} \mid \tau \times \tau \mid \tau + \tau \mid \tau \to \tau \mid \\ & \forall \alpha.\, \tau \mid \exists \alpha.\, \tau \mid \mu\, \alpha.\, \tau \mid \text{ref}\, \tau\end{aligned}$$

$$\begin{aligned}e \in \text{Expr} ::= {}& v \mid x \mid e_1(e_2) \mid \text{if } e \text{ then } e_1 \text{ else } e_2 \mid \text{fst}(e) \mid \text{snd}(e) \mid \text{ref}(e) \mid \\ & !\, e \mid e_1 \leftarrow e_2 \mid \text{fold}\, e \mid \text{unfold}\, e \mid \cdots \mid \text{flip}\end{aligned}$$

flip chooses uniformly between true and false.

Our implementation supports discrete uniform sampling as well

Probability distributions

A distribution over a countable type A is a non-negative map $\mu : A \to \mathbb{R}$ such that $\sum_{a \in A} \mu(a) \leq 1$.

Probability distributions have a monadic structure given by:

$$\mathrm{ret} : A \to \mathcal{D}(A)$$
$$\mathrm{ret}(a) \triangleq \lambda a'.\text{if } (a = a') \text{ then } 1 \text{ else } 0$$

$$\gg\!= : \mathcal{D}(A) \to (A \to \mathcal{D}(B)) \to \mathcal{D}(B)$$
$$(\mu \gg\!= f)(b) \triangleq \sum_{a \in A} \mu(a) \cdot f(a)(b)$$

Operational semantics

We start from a probabilistic head step reduction $\mathrm{hdStep}\colon \mathsf{Cfg} \to \mathcal{D}(\mathsf{Cfg})$:

$$(\lambda x.e)\ v, \sigma \to_{\mathsf{h}}^{1} e[v/x], \sigma$$

$$\dots$$

$$\mathsf{flip}, \sigma \to_{\mathsf{h}}^{1/2} b, \sigma \qquad b \in \{\mathsf{true}, \mathsf{false}\}$$

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and lift it to reduction in context $\mathrm{step}\colon \mathsf{Cfg} \to \mathcal{D}(\mathsf{Cfg})$:

$$\frac{e, \sigma \to_{\mathsf{h}}^{p} e', \sigma}{(K[e], \sigma) \to^{p} (K[e'], \sigma')}$$

Probabilistic evaluation

We define a “stratified” evaluation and full evaulation as the limit:

$$\mathrm{exec}_n^{\Downarrow}(e,\sigma) \triangleq \begin{cases} \mathrm{ret}\, e & \text{if } e \in \mathsf{Val} \\ \mathbf{0} & \text{if } e \in \mathsf{Val} \wedge n = 0 \\ \mathrm{step}(e,\sigma) \gg\!= \mathrm{exec}_m^{\Downarrow} & \text{if } e \in \mathsf{Val} \wedge n = m+1 \end{cases}$$

$$\mathrm{exec}^{\Downarrow}(e,\sigma) \triangleq \mathsf{lim}_{n\to\infty}\, \mathrm{exec}_n^{\Downarrow}$$

By summing over all values, we obtain the probability of termination:

$$\mathsf{Pterm}(e,\sigma) \triangleq \sum_{v \in \mathsf{Val}} \mathrm{exec}^{\Downarrow}(e,\sigma)(v)$$

Probabilistic languages in Iris

We define an abstract notion of probabilistic `Language`, in which `prim_step` is a function.

```
Structure language := Language {
  expr : Type;
  state : Type;
  (* ... *)
  prim_step : expr → state → distr (expr * state);
  (* ... *)
}.
```

We then lift it into an EctxLanguage

```
Structure ectxLanguage := EctxLanguage {
  (* ... *)
  fill : ectx → expr → expr;
  decomp : expr → ectx * expr;
  head_step : expr → state → distr (expr * state);
  (* ... *)
}.
```

and decompose expressions explicitly

```
Definition prim_step (e1 : expr Λ) (σ1 : state Λ)
    : distr (expr Λ * state Λ) :=
  let '(K, e1') := decomp e1 in
  '(e2, σ2) ← head_step e1' σ1; dret (fill K e2, σ2)
```

A coupling-based WP

Couplings 101

Couplings are a construction that allows us to reason relationally about probabilistic programs.

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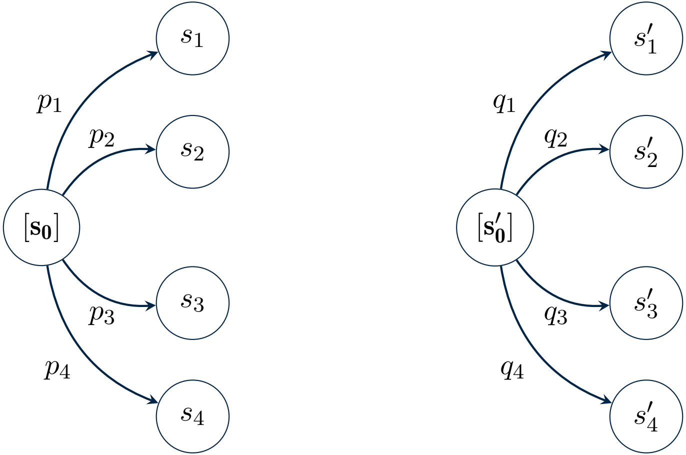
We then say that "μ_1 and μ_2 are coupled by R".

notation: $\mu_1 \sim \mu_2 : R$

Couplings 101



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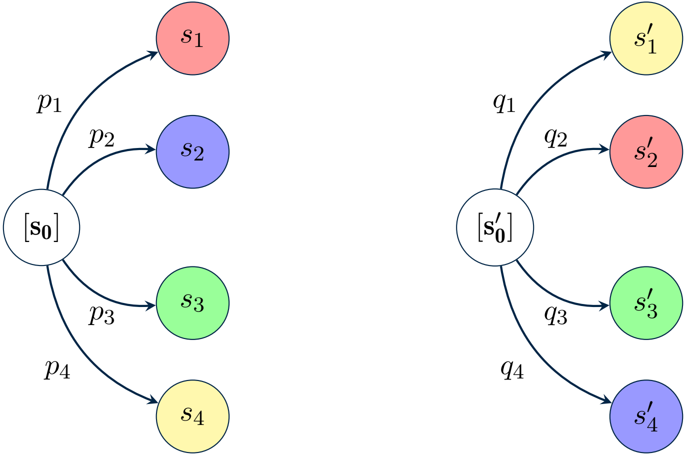

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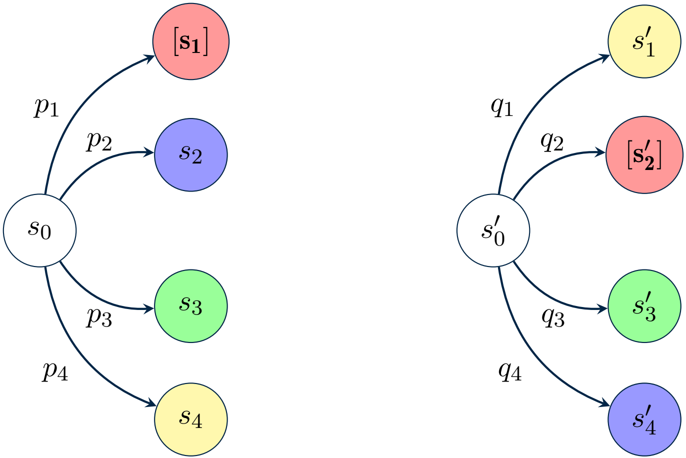

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Sequencing:

$$\frac{\mu_1 \sim \mu_2 : R \qquad \forall (a, b) \in R.\ f_1(a) \sim f_2(b) : S}{\mu_1 \gg\!= f_1 \sim \mu_2 \gg\!= f_2 : S}$$

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Elimination:

$$\frac{\mu_1 \sim \mu_2 : (=)}{\forall x. \mu_1(x) = \mu_2(x)}$$

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execCoupl couples every step on the LHS with 0 or more steps on the RHS.

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Abstraction through couplings

The coupling-based WP acts as an abstraction layer:

- The postcondition has type $\Phi : \mathsf{Val} \to \mathsf{iProp}$ (and not $\mathcal{D}(\mathsf{Val}) \to \mathsf{iProp}$)
- No explicit reasoning about probabilities, everything is hidden by execCoupl
- In fact, WP obeys the standard rules for the deterministic sequential fragment of **HeapLang**
- We only add new rules for probabilistic constructs

Adequacy

Assume:

$$\mathsf{specCtx} * \mathsf{spec}(e') \vdash \mathsf{wp}\ e\ \left\{v. \exists v'.\ \mathsf{spec}(v') * \varphi(v, v')\right\}$$

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- ReLoC: If e terminates, so does e', and the result values are related by φ
- Clutch: e terminates with lower or equal probability than e' and the result distributions are coupled by φ

Reasoning about contextual refinement

Contextual refinement

Two programs are contextually equivalent if they have the same observable behavior under any context.
We define contextual refinement through the termination probability:

$$\Theta \mid \Gamma \vdash e_1 \precsim_{\mathsf{ctx}} e_2 : \tau \triangleq \forall \tau', (\mathcal{C} : (\Theta \mid \Gamma \vdash \tau) \Rightarrow (\emptyset \mid \emptyset \vdash \tau')), \sigma.$$
$$\mathsf{Pterm}(\mathcal{C}[e_1], \sigma) \leq \mathsf{Pterm}(\mathcal{C}[e_2], \sigma)$$

Logical refinement

We define a ReLoC-style logic refinement and prove it sound wrt. contextual refinement:

$$\Delta \vDash_{\mathcal{E}} e_1 \precsim e_2 : \tau \triangleq \forall K.\, \mathsf{specCtx} \mathrel{-\!\!*} G(K[e_2]) \mathrel{-\!\!*} \mathsf{naTok}(\mathcal{E}) \mathrel{-\!\!*} \\ \mathsf{wp}\; e_1 \,\{v_1.\exists v_2.\, G(K[v_2]) * \mathsf{naTok}(\top) * [\![\tau]\!]_\Delta(v_1, v_2)\}$$

The value interpretation $[\![\tau]\!]_\Delta(v_1, v_2)$ is essentially the same as in ReLoC.

Some relational rules

We recover the standard ReLoC rules:

$$\frac{e_1 \overset{\text{pure}}{\rightsquigarrow} e_1' \qquad \triangleright(\Delta \vDash_{\mathcal{E}} K[e_1'] \precsim e_2 : \tau)}{\Delta \vDash_{\mathcal{E}} K[e_1] \precsim e_2 : \tau} \qquad \frac{e_2 \overset{\text{pure}}{\rightsquigarrow} e_2' \qquad \Delta \vDash_{\mathcal{E}} e_1 \precsim K[e_2'] : \tau}{\Delta \vDash_{\mathcal{E}} e_1 \precsim K[e_2] : \tau}$$

$$\frac{\forall \ell.\, \ell \mapsto v \twoheadrightarrow \Delta \vDash_{\mathcal{E}} K[\ell] \precsim e_2 : \tau}{\Delta \vDash_{\mathcal{E}} K[\mathsf{ref}(v)] \precsim e_2 : \tau} \qquad \frac{\ell \mapsto v \qquad \ell \mapsto w \twoheadrightarrow \Delta \vDash_{\mathcal{E}} K[()] \precsim e_2 : \tau}{\Delta \vDash_{\mathcal{E}} K[\ell \leftarrow w] \precsim e_2 : \tau}$$

A coupling rule

To reason relationally about probabilistic choices, the judgment satisfies

$$\frac{f : \mathbb{B} \to \mathbb{B} \text{ bijection} \qquad \forall b.\, \Delta \vDash_{\mathcal{E}} K[b] \precsim K'[f(b)] : \tau}{\Delta \vDash_{\mathcal{E}} K[\,\mathsf{flip}\,] \precsim K'[\,\mathsf{flip}\,] : \tau}$$

This rule builds a coupling for flip and sequences it through the contexts

Soundness

Theorem (Soundness)

Logical refinement implies contextual refinement

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In particular

$$\vDash e_1 \precsim e_2 : \mathsf{bool}$$

implies

$$\mathrm{exec}^{\Downarrow}(e_1, \sigma)(\mathsf{true}) \leq \mathrm{exec}^{\Downarrow}(e_2, \sigma)(\mathsf{true})$$
$$\mathrm{exec}^{\Downarrow}(e_1, \sigma)(\mathsf{false}) \leq \mathrm{exec}^{\Downarrow}(e_2, \sigma)(\mathsf{false})$$

But...

The approach fundamentally relies on being able to “synchronize” the probabilistic samplings.

$$\frac{f \text{ bijection} \qquad \forall b.\, \Delta \vDash_{\mathcal{E}} K[b] \precsim K'[f(b)] : \tau}{\Delta \vDash_{\mathcal{E}} K[\,\mathsf{flip}\,] \precsim K'[\,\mathsf{flip}\,] : \tau}$$

This is not always possible.

Eager vs Lazy sampling

$$\begin{aligned}\textsf{eager} &\triangleq \textsf{let } b = \textsf{flip in}\\ &\quad \lambda\_.\, b\end{aligned}$$

$$\begin{aligned}\textsf{lazy} \triangleq\ & \textsf{let } r = \textsf{ref}(\textsf{None}) \textsf{ in}\\ & \lambda\_.\ \textsf{match } !r \textsf{ with}\\ &\qquad \textsf{Some}(b) \Rightarrow b\\ &\quad\ \mid \textsf{None} \quad\ \Rightarrow \textsf{let } b = \textsf{flip in}\\ &\qquad\qquad\qquad\quad r \leftarrow \textsf{Some}(b);\\ &\qquad\qquad\qquad\quad b\\ &\quad\ \textsf{end}\end{aligned}$$

Eager vs Lazy sampling

$$\text{eager} \triangleq \text{let } b = \text{flip in} \\ \lambda\_.\, b$$

$$\begin{aligned}\text{lazy} \triangleq\ & \text{let } r = \text{ref}(\text{None}) \text{ in} \\ & \lambda\_.\ \text{match } !r \text{ with} \\ & \quad\ \ \text{Some}(b) \Rightarrow b \\ & \quad \mid \text{None} \quad\ \ \Rightarrow \text{let } b = \text{flip in} \\ & \qquad\qquad\qquad\quad r \leftarrow \text{Some}(b); \\ & \qquad\qquad\qquad\quad b \\ & \quad \text{end}\end{aligned}$$

How can we show $\vdash \text{eager} \simeq_{\text{ctx}} \text{lazy} : \text{unit} \to \text{nat}$?

Asynchronous probabilistic couplings

Asynchronous couplings

We extend the operational semantics with presampling tapes and labelled flips

$$
\begin{array}{ll}
\textsf{tape}, \sigma \to^{1} \iota, \sigma[\iota \mapsto \epsilon] & \text{if } \iota = \mathrm{fresh}(\sigma) \\
\textsf{flip}(\iota), \sigma \to^{1/2} n, \sigma & \text{if } \sigma(\iota) = \epsilon \\
\textsf{flip}(\iota), \sigma \to^{1} b, \sigma[\iota \mapsto \vec{b}] & \text{if } \sigma(\iota) = b \cdot \vec{b}
\end{array}
$$

Note: The tapes can only be populated at the logical level, no language operation writes to them

This is modelled by a "points-to-like" connective:

$$\iota \hookrightarrow \vec{b}$$

The asynchronous coupling rule looks like:

$$\frac{f \text{ bijection} \qquad e \notin \mathsf{Val} \qquad \iota \hookrightarrow \vec{b} \qquad \forall b.\, \iota \hookrightarrow \vec{b} \cdot b \;-\!\!*\; \Delta \vDash_{\mathcal{E}} e \precsim K'[f(b)] : \tau}{\Delta \vDash_{\mathcal{E}} e \precsim K'[\,\mathsf{flip}\,] : \tau}$$

Soundness relies on the fact that presampling does not change the result distribution

Future work

- Reasoning about approximate contextual equivalence.
- Supporting more general notions of probabilistic refinement, e.g. Markov Decision Processes.
- Supporting quantitative reasoning about expected costs, runtime, etc.
- Constructing couplings across recursive calls.

Clutch in a nutshell

- A probabilistic operational semantics for sequential probabilistic languages
- A unary coupling-based WP to prove relations between probabilistic programs
- A ReLoC-style logical relation to prove contextual refinement of probabilistic programs
- A ghost resource to reason about samples that happen in the future

Try Clutch! `https://github.com/logsem/clutch`